Research Essay – [redacted]

updated November 1, 2017

My research lies in the area of Partial Differential Equations, and much of my work has been dedicated to investigating some fundamental properties of solutions to equations that enjoy a combination of nonlocal interactions with a structure condition called "ellipticity" (elaborated below). For many phenomena and equations, there is an either explicit or implicit driving feature that takes the form of an integro-differential operator. That is to say, for an unknown function on $\mathbb{R}^d$, u, one studies the operator,

$$L(u,x) = \int_{\mathbb{R}^d} \left(u(x+h) - u(x)\right) K(x,h)dh, \tag{1}$$

and a given function, f, in some set, Ω, one seeks to know various consequences of the equation

$$L(u,x) = f(x) \text{ in } \Omega. \tag{2}$$

This operator, L, is simply a weighted average of the differences of the unknown function, u, weighted by the non-negative function, $K(x,h) \geq 0$. In many situations K may be singular at $h = 0$, and so a proper interpretation of this integral is required, but for the sake of presentation, we leave it as it is in (1). Operators such as these integro-differential L are fundamental to many areas of mathematics, they have seen a huge surge in activity over the past 2 decades or so, and much of my research is related to them. Lately my research has led me to find connections between some well known problems that initially have no obvious link with operators like L, and then set up a scenario where recent tools for the study of L can be applied to these older problems within a new context.

Operators such as L appear in a myriad of contexts, some in a straightforward fashion, and some as a hidden feature. Two situations that are very similar mathematically and appear as a straightforward application of the integro-differential theory would be: the modeling of insects or seeds moving on relatively long space-time scales; and the modeling of tracer particles flowing through an aquifer. In both of these situations, nonlocality (the fact that (1) sees the global values of u) appears because although locally these particles may be undergoing a diffusion, they are typically caught up in a different random process that manifests itself on longer space and time scales. For example, it could be wind patters for the insects or regions of high flow rates in an aquifer (e.g. cracks in an otherwise dense configuration of slow flow). The relatively longer time scales (that are relevant to our observations) show these agents to be jumping from point to point, even though on the microscopic time scale their motion is continuous. These apparent jumps in space will appear in the time-infinitesimal description of the motion of the particles, by making it necessary to average over all possible next jump locations of these particles, and this is why $u(x+h) - u(x)$ must be averaged over all of $\mathbb{R}^d$. In this case, u can be considered the concentration of the particles being followed. An example of where an integro-differential equation appears in a less obvious fashion is, for example the case of the motion of an interface between two different fluids, sometimes arising in porous media flow or a Hele-Shaw type flow. These dynamics are governed by quasi-static behavior where inside the region of each phase, a potential develops nearly instantaneously, and then the movement of the interface between the two phases is governed by a velocity field based on the gradient of the potentials, via a balance law relating the relative strength of each gradient along the boundary. The new configuration of the boundary of the phases induces a new potential instantaneously, and hence a new gradient induces further motion, continuing to drive the boundary via these dynamics. It turns out that this free boundary between the phases can be shown to evolve via a parabolic differential equation driven by an operator of the form (1), and also this parabolic flow is inherently *nonlinear*. That such Hele-Shaw type flows (sometimes called a Muskat problem) had a connection with a nonlocal operator is not new, however, it is new that under an appropriate interpretation, the flow is of nonlinear parabolic type, with an operator like (1) (i.e. a nonlinear and integro-differential analog of the heat equation). This is a topic of ongoing work with my collaborators, [redacted] and [redacted] [1]. Once a connection to operators like (1) is established, a whole new set of tools is available, and in particular there are many substantial developments in this integro-differential toolbox over the past 20 years. In ongoing work, my collaborators and I plan to show new regularity results for free boundary problems in this family by invoking recent ideas and results from the arena of integro-differential equations like (1).

What are the features of L that make the equation (2) impose a structure on its solutions? Broadly, one could call these features *ellipticity*. We have already assumed that $K(x,h) \geq 0$, and this gives a feature that we call the global comparison property:

$$\text{if for all } y,\ u(y) \leq v(y) \text{ and for } x_0,\ u(x_0) = v(x_0), \text{ then } L(u,x_0) \leq L(v,x_0). \qquad (3)$$

(Note, this is just an extension of the fact we teach to calculus students that at the point of a local maximum of a C^2 function, u, one sees that $u'' \leq 0$ at the local max.) Now, what if, on top of this, one assumes some non-degeneracy of K, in the sense that for a large portion of h near to $h = 0$, you have that $K(x,h) \geq |h|^{-d-\alpha}$. This non-degeneracy is what we call *uniform ellipticity*. What it means from a practical point of view is that in order that the equation (2) holds in any reasonable sense, the function, u must in fact be quite well behaved. This is because a singularity of the form $|h|^{-d-\alpha}$ is not integrable at $h = 0$, and so cancellation effects from u must be present to account for the fact that one has that, e.g. $L(u,x)$ is bounded (i.e. when f is bounded and continuous). The actual sitation is, of course, more nuanced due to at least the fact that usually equation (2) cannot be interpreted in a classical sense – only in a weak sense. Nonetheless, this intuition persists, and it is manifested by results that show u must have very nice properties. Figuring out how well behaved such u must be is usually described as *regularity theory* for equations like (2). A significant portion of my recent works have taken on various aspects of this pursuit, and they are: [3], [8], [9], [10]. I would like to point out that my work with Silvestre, [10], is the most advanced result for these types of equations and questions, and it contains nearly every other elliptic (and also parabolic) regularity result for integro-differential equations as a sub-case.

Let's go back to another example of where operators like (1) may arise. One can consider tracking a particle undergoing a diffusion (or for simplicity, just a random walk) inside, say, a glass of water such that the particle reflects off the boundary of the container every time it comes in contact with the boundary. Now, suppose that an observer cannot witness the location of the particle during its whole trajectory, but rather only when the particle hits the boundary of the container. In this situation, the hidden diffusion inside the container would cause them to witness the particle jumping discretely and randomly around the boundary. Under an appropriate time rescaling (required because diffusions spend zero time on a fixed, lower dimensional set), this is a new random process in its own right, and it is sometimes called the boundary process, which is a type of "jump-diffusion". In the simplest case that the "glass of water" is the upper-half space, $\mathbb{R}^{d+1}_+$, then the operator that governs the motion of this boundary process is of the form (1), and actually in this simplified setting the operator, L, becomes the 1/2-Laplacian. This is a long-known, and classical result. In the case that the "glass" of water is actually a glass of water, with the boundary of the container being a nice manifold, then the validity of (1) as the operator that governs the boundary process was proved decades ago by [redacted] [7] (but the integral in (1) is now over the boundary manifold). Since that time, not much has been said about the PDE properties of these boundary processes.

In terms of PDE and Analysis, the phenomenon described here is governed by the operator that is called the Dirichlet-to-Neumann operator. In this simplest case of true Brownian diffusion (or a balanced random walk), this means that the boundary process is governed by the operator, let's call it I, that is defined as

$$\phi \text{ is given, and one solves the equation } \Delta U_\phi = 0 \text{ in } \Omega, \text{ subject to } U_\phi = \phi \text{ on } \partial\Omega,$$

$$\text{and } I \text{ is constructed as } I(\phi, x) = \partial_n U_\phi(x).$$

Here, Ω would be the container of water, $\partial\Omega$ is its boundary, and $\partial_n U_\phi$ is the inward normal derivative of U_ϕ on $\partial\Omega$. We point out that so far, this I is a *linear* operator, and furthermore, the curious reader can check that it also satisfies the global comparison property (3) for functions on $\partial\Omega$ thanks to the order preserving nature of solutions of Laplace's equation and our choice to use the inward normal vector.

In order to make this example slightly more interesting, we will add a layer of complexity. Suppose now there is still the observer who can only see the particle when it is on the boundary, but also we introduce a player who gets to modify the particle's dynamics while it is inside the container. One can think that instead of water, there is a special medium that at every point in time and every point in space, a player can choose to make a modification to the "water" to effect its diffusive properties, possibly picking one preferential axis over another, but still required to allow a diffusive behavior, just not necessarily one that is rotationally and spatially homogeneous. What this means is that the player inside causes the particle to undergo a *nonlinear* diffusion. In practical terms, this means that above, instead of solving $\Delta U_\phi = 0$, one would solve $F(D^2 U_\phi) = 0$ for some uniformly elliptic operator, F; hence the corresponding Dirichlet-to-Neumann operator, I, is also *nonlinear*. Until some recent work by [redacted] and myself in [5] and by [redacted] and myself in [2], it was not known if such a nice representation of these nonlinear versions of I in terms of (1) was valid. It turns out for these I, a representation as in (1) is valid *provided* one allows for a min-max procedure over a set of linear operators that appear in (1)– hence taking into account the nonlinearity in this situation. We went further to show that not only does a representation like (1) hold for I, but in the nonlinear setting, one is forced to confront measures, "$K(x, h)dh$", that are not necessarily absolutely continuous with respect to surface measure and possess properties that are different from any other integro-differential operators that have been studied so far (to the best of my knowledge).

Finally, in this scenario of a boundary process, if we insert a third person into the mix, we can make a connection with homogenization. Suppose now that there are the observer, the player inside, and also a player on the boundary of the container. Suppose this player on the boundary is allowed to give the particle a cost (or payout) at every time and location where it happens to hit the boundary. Suppose also that this player structures these payouts with a highly oscillatory structure over very short space scales relative to the outside observer. Will the outside observer see the same thing as before, or will the dynamics look different from the original situation? Answering questions like these fall into the realm of boundary homogenization, and this is a field that has commanded a lot of attention from many researchers lately. I just mention that two of my recent works with [redacted] address questions like these, and those are [6] and [4]. It turns out that the result in [4] is the only one that can handle the situation in which the diffusion inside the container also contains a drift term that can be increasingly large as the length scale of oscillations goes to zero.

References

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Teaching Philosophy – [redacted]. updated October 31, 2017

This is my philosophy as it pertains to teaching mathematics at a crucial stage in undergraduate development, the introduction to mathematical logic and proofs. At MSU, this class is MTH 299, Transitions to Higher Mathematics. My first day motivational speech is, as said to the students: "Don't take this the wrong way, and this is certainly not your fault– it is our fault– you have just finished approximately 12 years of being taught very poor mathematical habits. It is my goal to break down your existing mathematical reasoning, expectations, and habits, down to the ground, and we will build them back up again, together. I will be your coach, not your instructor." Why do I say these things? How do I connect with these students to achieve their promised growth? On one hand, it's simple, but in reality it can be painfully complex.

Let's start with some of **my basic beliefs:** mathematics is everywhere; critical thinking and problem solving are useful in nearly every aspect of life; sadly, many young people demonstrate a fear of mathematics that almost seems to be the result of a strange irrational social contagion. To this foundation of my teaching system, we can add the next layers of **my personal interaction with mathematics:** mathematics is hard; I find challenging things fun (for the most part); my understanding / research does not happen quickly; I make many mistakes initially; I love learning from my mistakes; perseverance is comforting and ultimately rewarding. Next, **what about my target audience?** Regardless of their actual academic background, many of these students show up with a seemingly stunted mathematical growth, somehow stuck at the level of wanting to get a quick answer, collect some points, click a button, get an electronic "high five", and move on. Some other things I've observed are: obsession with quick/numerical answers; instant gratification; repetition; fear/refusal to explore ideas; inability to link ideas/problem solving across multiple genres; no appreciation for incorrect arguments; inability to write/explain mathematical reasoning. Obviously, this mentality is deathly for their success in upper division mathematics classes, and it is far from the types of strengths we want them to have: problem solving; independence; patience; perseverance; abstract thinking; logical presentation; clear proof writing. It is my job to fix this state of affairs; in a nutshell, I help my students adopt some aspects of my own personal mathematical experiences to help them navigate *their own path* to success.

In my opinion, theoretically, the solution is simple: give the students a safe, productive space, without judgement, with lots of constructive input, and lots of supervised time to practice and fail repeatedly, to allow them to build their own understanding and abilities of problem solving, *on their own terms*, in order to gain future succes. In buzzwords, this is my take on *active learning*, or a *flipped classroom.* The importance of this environment cannot be overstated; it is fundamental to each student building understanding and proficiency organically, resulting in more deeply rooted and longer lasting mastery. So, what's the problem, this should be easy, right? In practice, it's not, but rather more like chasing a unicorn. Because I am more interested in teaching my students a viewpoint and skills, rather than a long list of facts and formulas, my classroom and material has to be dynamic and versatile. With the help of the other instructors I have mentored, we have created materials that can be modified quickly, on the fly, from week to week, but also in the span of a class meeting. Hence, this dynamic approach is, *by definition*, easily modifiable. I have a list of indicators, like performance on homework, exams, and competency in group discussions, which allow me to tune the next steps based on students' current status, making it to the finish line at the end of the semester.

Quite frankly, it is us, as mathematicians who get in the way of our own teaching success. The hardest part for myself and my peers to realize and overcome on a daily basis is that these students *don't* think like us! It is so hard to knowingly help a student go down a path that is not quite the best, but nonetheless is a path towards one possible solution. To nurture this diversity in problem solving in a constructive way that resonates with the students is the hardest part. It's like a chef who is expected to make a world class dinner every night, from ingredients over which they have very little control. For the best chefs, this is OK, but for the rest of us, it is painstaking. I take great pride that after 2.5 years of experimentation, I have obtained some success with my students in this MTH 299 program/experiment.